Forcing with filters and ideals (part III.) Malykhin's Problem

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Basics

- An ideal $\mathcal{I}$ on ω is *ω-hitting* if for every $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$ there is an $I \in \mathcal{I}$ such that $A_n \cap I$ is infinite for all $n \in \omega$,
- If you split an ω-hitting ideal into countably many pieces, one of the pieces is ω-hitting.
- (Katětov order)Let $\mathcal{I}$ and $\mathcal{J}$. $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$.

$$\mathbb{L}_{\mathcal{F}} = \{ T \subseteq \omega^{<\omega} : T \text{ is a tree with stem } s_T \text{ such that}$$
$$\text{for all } t \in T, t \supseteq s_T \Rightarrow succ_T(t) \in \mathcal{F}\},$$

ordered by inclusion.

$succ_T(t) = \{n \in \omega : t^\frown n \in T\}$,

Definition

Given $s \in \omega^{<\omega}$ and φ formula in the forcing language we say that s *favours* φ if no condition in $\mathbb{L}_{\mathcal{F}}$ with stem s forces "$\neg\varphi$".

Preservation of ω-hitting

Definition

A forcing notion $\mathbb{P}$ *strongly preserves ω-hitting* if for every sequence $\langle \dot{A}_n \colon n \in \omega \rangle$ of $\mathbb{P}$-names for infinite subsets of ω there is a $\langle B_n \colon n \in \omega \rangle$ sequence of infinite subsets of ω such that for any $B \in [\omega]^\omega$, if $B \cap B_n$ is infinite for all n then $\Vdash_{\mathbb{P}}$ "$B \cap \dot{A}_n$ is infinite for all n".

Proposition 1. (Brendle-H.)

Finite support iteration of forcings strongly preserving ω-hitting strongly preserves ω-hitting.

Lemma 2. (Brendle-H.)

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F} = \mathcal{I}^*$ be the dual filter. Then the following are equivalent:

(1) For every $A \in \mathcal{I}^+$ and every $\mathcal{J} \leqslant_K \mathcal{I} \restriction A$ the ideal $\mathcal{J}$ is not ω-hitting,

(2) $\mathbb{L}_{\mathcal{F}}$ strongly preserves ω-hitting, and

(3) $\mathbb{L}_{\mathcal{F}}$ preserves ω-hitting.

Malykhin

Theorem (H.-Ramos García)

It is consistent with **ZFC** that every separable Fréchet group is metrizable.

Plan of the proof:
Let $V \models$ CH and $\diamond(S^{\omega_2}_{\omega_1})$. Construct a FS iteration of length ω_2 σ-centered forcing notions, eventually taking care of all countable Fréchet groups of weight ω_1. At stage α when dealing with the group $\mathbb{G}_\alpha$ handed to us by a bookkeeping device we need to do three things:

1. add a set $A \subseteq \mathbb{G}_\alpha$ which has the neutral element $1_\mathbb{G}$ as an accumulation point, and does not have a sequence converging to$1_\mathbb{G}$,
2. make sure that we do not add convergent sequences to the sets added earlier in the iteration.
3. make sure that $1_\mathbb{G}$ remains in the closure of A later on.

Fréchet idealized

- Given a space X and a point $x \in X$ we denote by $\mathcal{I}_x$ the dual ideal to the filter of neighbourhoods of x, $\mathcal{I}_x = \{A \subseteq X : x \notin \bar{A}\}$.
- If X is countable then the infinite members of $\mathcal{I}^\perp = \{J \subseteq X : (\forall I \in \mathcal{I})|I \cap J| < \omega\}$ are exactly the sequences convergent to x.
- The space X is Fréchet at x iff every $\mathcal{I}_x$-positive set contains an infinite element of $\mathcal{I}_x^\perp$ iff $\mathcal{I}_x^{\perp\perp} = \mathcal{I}_x$ iff for no $A \in \mathcal{I}_x^+$ is the ideal $\mathcal{I}_x \restriction A$ tall.

Definition

A forcing notion $\mathbb{P}$ *seals* an ideal $\mathcal{I}$ if it adds an $\mathcal{I}$-positive set A such that the ideal $\mathcal{I} \restriction A$ is countably tall.

Sealing $\mathcal{I}$ by $\mathbb{L}_{\mathcal{F}}$

Lemma 3.

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F}$ be al filter on ω.

IF

$\mathcal{I} \cap \mathcal{F} = \emptyset$ and for every countable family $\mathcal{H} \subseteq \mathcal{F}^+$ there is an $I \in \mathcal{I}$ such that $H \cap I \in \mathcal{F}^+$ for all $H \in \mathcal{H}$ (i.e. $\mathcal{I}$ is *ω-hitting mod $\mathcal{F}$*)

THEN

the forcing $\mathbb{L}_{\mathcal{F}}$ seals the ideal $\mathcal{I}$.

As $\mathcal{I} \cap \mathcal{F} = \emptyset$, $\mathcal{I}^* \subseteq \mathcal{F}^+$ and by genericity
$\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$X \cap \dot{A}_{gen}$ is infinite for all $X \in \mathcal{F}^+$". Thus, $\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$\dot{A}_{gen} \in \mathcal{I}^+$".

We are proving...

$\mathcal{I}$ is ω-hitting mod $\mathcal{F}$ implies that $\mathbb{L}_{\mathcal{F}}$ seals the ideal $\mathcal{I}$.

Let $\langle \dot{A}_n \colon n \in \omega \rangle$ be a sequence of $\mathbb{L}_{\mathcal{F}}$-names for infinite subsets of $\dot{A}_{gen}$. Assume that for all $I \in \mathcal{I}$ there are $T_I \in \mathbb{L}_{\mathcal{F}}$, and natural numbers n_I, m_I such that $T_I \Vdash$ "$\dot{A}_{n_I} \cap I \subseteq m_I$". $(\star)$ Let
$\mathrm{rk}_n(s) = 0$ if $\exists B \in \mathcal{F}^+ \forall b \in B (s^\frown b$ favors $b \in \dot{A}_n)$ and
$\mathrm{rk}_n(s) \leqslant \alpha$ if $\exists B \in \mathcal{F}^+ \forall b \in B (\mathrm{rk}_n(s^\frown b) < \alpha)$ for $\alpha > 0$.

Claim. $\mathrm{rk}_n(s) < \infty$ for all s and n.

Fix n. Let $k \in \omega$ and let $\rho_k(s) = 0$ if $\exists b > k(s$ favors $b \in \dot{A}_n)$

Then $\rho_k(s) < \infty$ (easy). Note that s only favours elements of $\mathrm{ran}(s)$ ($\dot{A}_n$ is forced to be a subset of $\dot{A}_{gen}$).
If $\rho_k(s) = 1$ then $\mathrm{rk}_n(s) = 0$.
$(\exists B \in \mathcal{F}^+ \forall \in B$ $s^\frown b$ favors $a \in \dot{A}_n$ for some $a = a_b$ with $a > k$. As $\rho_k(s) > 0$, $\{b : a = a_b\} \in \mathcal{F}^*$ for each a and as $a_b \in \mathrm{ran}(s) \cup \{b\}$, on an $\mathcal{F}$-positive set, $a_b = b$.)
Now, let $k > \max(\mathrm{ran}(s))$. Then $\rho_k(s) \geqslant 1$, hence $\mathrm{rk}_n(s) < \infty$.

... and we prove.

$\mathcal{I}$ is ω-hitting mod $\mathcal{F}$ implies that $\mathbb{L}_\mathcal{F}$ seals the ideal $\mathcal{I}$.

$\mathrm{rk}_n(s) = 0$ if $\exists B_{s,n} \in \mathcal{F}^+ \, \forall b \in B_{s,n} (s^\frown b$ favors $b \in \dot{A}_n)$ and $\mathrm{rk}_n(s) < \infty$ for all s and n.

WLOG $\mathrm{rk}_{n_I}(s_{T_I}) = 0$ for all $I \in \mathcal{I}$.

For every $I \in \mathcal{I}$ consider B_{s_I,n_I}. Since $\mathcal{I}$ is ω-hitting mod $\mathcal{F}$, there is an I which intersects all the $B_{s,n}$'s in a $\mathcal{F}$-positive set, and i.p. $I \cap B_{s_I,n_I} \in \mathcal{F}^+$. So, there is a $b > m_I$ with $b \in I \cap B_{s_I,n_I} \cap \mathrm{succ}_{T_I}(s_I)$. As $s_I^\frown b$ favors $b \in \dot{A}_{n_I}$, there is a $T \leqslant T_I$ whose stem extends $s_I^\frown b$ such that $T \Vdash$ "$b \in \dot{A}_{n_I}$", a contradiction.

A lemma of Barman and Dow

Lemma 4. (Barman-Dow)

Let X be a Fréchet space without isolated points, let $x \in X$ and let $\mathcal{M}$ be a countable family of nowhere dense subsets of X. Then there is an infinite sequence C_x converging to x with only finite intersection with every element of $\mathcal{M}$.

Proof. Fix $\langle M_n \colon n \in \omega \rangle$ an enumeration of $\mathcal{M}$ and a sequence $\langle x_n \colon n \in \omega \rangle \subseteq X \setminus \{x\}$ converging to x. Let $X_n = X \setminus \left(\{x\} \cup \bigcup_{i<n} M_i\right)$. Then X_n is dense in X for every n. Since X is Fréchet, there is a sequence $\langle x^n_k \colon k \in \omega \rangle \subseteq X_n$ converging to x_n for each $n \in \omega$. Put $X' = \{x^n_k \colon k, n \in \omega\}$. Then $x \in \overline{X'}$, and hence there is a sequence $C_x \subseteq X'$ converging to x. By the construction, the sequence C_x is as required.

Main lemma

Proposition 4.

Let $X = (\omega, \tau)$ be a regular Fréchet space, $x \in X$ be such that $\pi\chi(x, X) > \omega$. Let $\mathcal{G}$ be the filter of dense open subsets of X. Then:

(1) $\mathbb{L}_{\mathcal{G}}$ seals $\mathcal{I}_x$, and

(2) $\mathbb{L}_{\mathcal{G}}$ strongly preserves ω-hitting.

Recall

Lemma 3.

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F}$ be al filter on ω.
$\mathcal{I}$ is ω-hitting mod $\mathcal{F}$ $\Rightarrow$ $\mathbb{L}_{\mathcal{F}}$ seals the ideal $\mathcal{I}$.

and

Lemma 2.

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F} = \mathcal{I}^*$ be the dual filter. Then:
$\mathbb{L}_{\mathcal{F}}$ strongly preserves ω-hitting $\Leftrightarrow$ For every $A \in \mathcal{I}^+$ and every $\mathcal{J} \leqslant_K \mathcal{I} \restriction A$ the ideal $\mathcal{J}$ is not ω-hitting.

Michael, use the blackboard!

Checking back with the plan...

Plan of the proof:
Let $V \models$ CH and $\diamond(S^{\omega_2}_{\omega_1})$. Construct a FS iteration of length ω_2 σ-centered forcing notions, eventually taking care of all countable Fréchet groups of weight ω_1. At stage α when dealing with the group $\mathbb{G}_\alpha$ handed to us by a bookkeeping device we need to do three things:

1. add a set $A \subseteq \mathbb{G}_\alpha$ which has the neutral element $1_{\mathbb{G}}$ as an accumulation point, and does not have a sequence converging to $1_{\mathbb{G}}$, (Done! Main lemma)
2. make sure that we do not add convergent sequences to the sets added earlier in the iteration. (Done! Preservation of ω-hitting by $\mathbb{L}_{nwd^*}$ and FSI)
3. make sure that $1_{\mathbb{G}}$ remains in the closure of A later on.

Finally algebra...

Definition

Let $(\mathbb{G}, \cdot)$ be an abstract group and let $A \subseteq \mathbb{G} \setminus \{1_\mathbb{G}\}$. A subset B of $\mathbb{G}$ is called *A-large* if for every $b \in A$ and $a \in \mathbb{G}$, either $a \in B$ or $b \cdot a^{-1} \in B$.

The intention is as follows: Let A_{gen} be the generically added set. If there was a finer group topology in which $1_\mathbb{G}$ is not in the closure of X, then there would be a set $U \subseteq \mathbb{G}$ containing $1_\mathbb{G}$ and such that $U \cdot U \cap A_{gen} = \emptyset$, a new open neighbourhood of $1_\mathbb{G}$ disjoint from A_{gen}. A set B is A_{gen}-large iff it contains the complement of such a U.

Definition

A family $\mathcal{C}$ of subsets of a group $\mathbb{G}$ is *ω-hitting w.r.t* A if given $\langle A_n \colon n \in \omega\rangle \subset A$-large there is a $C \in \mathcal{C}$ s. t. $C \cap A_n$ is infinite for all n.

Algebra and convergent sequences save the day!

Again, suppose that a set U is a new open neighbourhood of $1_{\mathbb{G}}$ disjoint from A_{gen} (such that $U \cdot U \cap A_{gen} = \emptyset$). The information about the future group topologies extending our topology τ, includes information about **convergent sequences**: If C converges to $1_{\mathbb{G}}$ in τ it will also converge in all future topologies, in particular, C would have to be almost contained in U!

However, if we manage to prove that for every candidate for U there is a C which is not almost contained in U (equivalently, has an infinite intersection with the A_{gen}-large set which is the complement of U) and also manage to preserve this, we will be able to show that $1_{\mathbb{G}}$ will be in the closure of X in all future group topologies extending τ!!

Definition

We say that a relation $R \subseteq \mathbb{G} \times \mathbb{G}$ is *large* if for every $b, a \in \mathbb{G}$, either $\langle b, a \rangle \in R$ or $\langle b, b \cdot a^{-1} \rangle \in R$. By large we will denote the family of relations which are large.

An algebraicly-topological lemma

Lemma 5.

Let $\mathbb{G}$ be a countable Fréchet group and let $\langle R_n \colon n \in \omega \rangle$ be a sequence of large relations. Then there is a sequence C convergent to $1_{\mathbb{G}}$ such that $R_n^{-1}[C \setminus F] \in \mathrm{nwd}(\mathbb{G})^+$ for every $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$.

For every $n \in \omega$ let $Z_n = \{a \in \mathbb{G} \colon R_n^{-1}(a) \in \mathrm{nwd}(\mathbb{G})\}$ and put
$\mathcal{M} = \{R_n^{-1}(a) \cdot a^{-1} \colon a \in Z_n, n \in \omega\} \cup \{Z_n \colon Z_n \in \mathrm{nwd}(\mathbb{G})\}$.
Let $C \to 1_{\mathbb{G}}$ be such that $|C \cap M| < \omega$ for all $M \in \mathcal{M}$.

Claim: $R_n^{-1}[C \setminus F] \in \mathrm{nwd}(\mathbb{G})^+$ for every $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$.

Let $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$ are given.

Case 1. $Z_n \in \mathrm{nwd}(\mathbb{G})$.

Then there is an $a \in C \setminus F$ such that $R_n^{-1}(a) \in \mathrm{nwd}(\mathbb{G})^+$.

Case 2. $Z_n \in \mathrm{nwd}(\mathbb{G})^+$.

Fix $a \in Z_n$. Then $b \notin R_n^{-1}(a) \cdot a^{-1}$ eq. $b \cdot a \notin R_n^{-1}(a))$ for almost all $b \in C$. Since R_n is large, $\langle b \cdot a, b \rangle \in R_n$ for almost all $b \in C$.
I.p., $\{b \cdot a \colon b \in C\} \subseteq^* R_n^{-1}[C \setminus F]$ and $\{b \cdot a \colon b \in C\}$ converges to a.
Thus, $Z_n \subseteq \overline{R_n^{-1}[C \setminus F]}$ and hence also $R_n^{-1}[C \setminus F] \in \mathrm{nwd}(\mathbb{G})^+$.

Preserving A_{gen} has $1_{\mathbb{G}}$ in its closure

Lemma 6.

Let $\mathbb{G}$ be a countable Fréchet topological group. Then

$$\Vdash_{\mathbb{L}_{\mathrm{nwd}^*(\mathbb{G})}} \text{“}\mathcal{C} \text{ is } \omega\text{-hitting w.r.t.. } \dot{A}_{gen}\text{”},$$

where $\mathcal{C} = \mathcal{I}^{\perp}_{1_{\mathbb{G}}}$ is the ideal consisting of sequences converging to $1_{\mathbb{G}}$.

If not, then there are a sequence $\langle \dot{A}_n \colon n \in \omega \rangle$ of $\mathbb{L}_{\mathrm{nwd}^*(\mathbb{G})}$-names and a condition $T^* \in \mathbb{L}_{\mathrm{nwd}^*(\mathbb{G})}$ such that $T^* \Vdash$ “$\forall n \in \omega \, (\dot{A}_n \in \dot{A}_{gen}$-large)” and for every $C \in \mathcal{C}$ there are a condition $T_C \in \mathbb{L}_{\mathrm{nwd}^*(\mathbb{G})}$, a natural number n_C and F_C a finite subset of $\mathbb{G}$ such that $T_C \Vdash$ “$C \cap \dot{A}_{n_C} \subseteq F_C$”. $(\star)$
For each $s \in T^*$ with $s \supseteq s_{T^*}$ and each natural number n, put

$$R_{s,n} = \{\langle b, a \rangle \colon b \in \mathrm{succ}_{T^*}(s) \Rightarrow s^\frown b \text{ favors } a \in \dot{A}_n\}.$$

Claim. The relation $R_{s,n}$ is large.

Preserving A_{gen} has $1_{\mathbb{G}}$ in its closure

we are proving

Let $\mathbb{G}$ be a countable Fréchet topological group. Then

$$\Vdash_{\mathbb{L}_{\mathrm{nwd}^*(\mathbb{G})}} \text{"}\mathcal{C} \text{ is } \omega\text{-hitting w.r.t.. } \dot{A}_{gen}\text{"},$$

where $\mathcal{C} = \mathcal{I}^{\perp}_{1_{\mathbb{G}}}$ is the ideal consisting of sequences converging to $1_{\mathbb{G}}$.

For each $s \in T^*$ with $s \supseteq s_{T^*}$ and each natural number n, put

$$R_{s,n} = \{\langle b, a\rangle \colon b \in \mathrm{succ}_{T^*}(s) \Rightarrow s^\frown b \text{ favors } a \in \dot{A}_n\}.$$

Claim. The relation $R_{s,n}$ is large.
By previous lemma, there is a $C \in \mathcal{C}$ such that $R^{-1}_{s,n}(C \setminus F) \in \mathrm{nwd}(\mathbb{G})^+$ for every $s \in T^*$ with $s \supseteq s_{T^*}$, $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$. In particular, $R^{-1}_{s_C,n_C}(C \setminus F_C) \in \mathrm{nwd}(\mathbb{G})^+$, where $s_C = s_{T_C}$. Pick an $b \in \mathrm{succ}_{T_C}(s_C) \cap R^{-1}_{s_C,n_C}(C \setminus F_C)$. Then, there is an $a \in C \setminus F_C$ such that $s_C^\frown b$ favors $a \in \dot{A}_{n_C}$, and hence there is a condition $T \leqslant T_C$ whose stem extends $s_C^\frown b$ such that $T \Vdash$ "$a \in \dot{A}_{n_C}$", a contradiction to the initial assumption $(\star)$.

One last bit

A forcing $\mathbb{P}$ *strongly preserves ω-hitting w.r.t. A* if for every $\dot{B}$ a $\mathbb{P}$-name for a A-large subset of a group $\mathbb{G}$ there is a sequence $\langle B_n \colon n \in \omega \rangle \subset A$-large such that for any $C \subseteq \mathbb{G}$, if $C \cap B_n$ is infinite for all n then $\Vdash_{\mathbb{P}}$ "$C \cap \dot{B}$ is infinite".

Lemma 7.

Let $\mathbb{P}$ be a σ-centered forcing notion. Then $\mathbb{P}$ strongly preserves ω-hitting w.r.t. A.

Proof. WLOG, $\mathbb{P}$ is a complete Boolean algebra, $\mathbb{P}^+ = \bigcup_{n \in \omega} \mathcal{U}_n$, all $\mathcal{U}_n$ being ultrafilters. Let $\dot{B}$ be a $\mathbb{P}$-name for an A-large subset of a group $\mathbb{G}$. For every $n \in \omega$, put

$$B_n = \{a \in \mathbb{G} : [\![a \in \dot{B}]\!] \in \mathcal{U}_n\}.$$

Each B_n is A-large and if C is a subset of $\mathbb{G}$ such that $C \cap A_n$ is infinite for all n, then $\Vdash_{\mathbb{P}}$ "$C \cap \dot{A}$ is infinite".

Lemma 8.

Finite support iteration of ccc forcings strongly preserving ω-hitting w.r.t. A strongly preserves ω-hitting w.r.t. A.

Last check of the plan

Plan of the proof:
Let $V \models \mathsf{CH}$ and $\diamond(S^{\omega_2}_{\omega_1})$. Construct a FS iteration of length ω_2 σ-centered forcing notions, eventually taking care of all countable Fréchet groups of weight ω_1. At stage α when dealing with the group $\mathbb{G}_\alpha$ handed to us by a bookkeeping device we need to do three things:

1. add a set $A \subseteq \mathbb{G}_\alpha$ which has the neutral element $1_{\mathbb{G}}$ as an accumulation point, and does not have a sequence converging to $1_{\mathbb{G}}$, (Done! Main lemma)
2. make sure that we do not add convergent sequences to the sets added earlier in the iteration. (Done! Preservation of ω-hitting by $\mathbb{L}_{nwd^*}$ and FSI)
3. make sure that $1_{\mathbb{G}}$ remains in the closure of A later on. (Done! by Lemma 6 and preservation of ω-hitting w.r.t. A).

Uffffffffffff!!

THANK YOU!!